The Strong Perfect Graph Theorem

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1 Introduction

In this note, all graphs are simple (no loops or multiple edges) and finite. The vertex set of graph G is denoted by $V(G)$ and its edge set by $E(G)$. A *stable set* is a set of vertices no two of which are adjacent. A *clique* is a set of vertices every pair of which are adjacent. The cardinality of a largest clique in graph G is denoted by $\omega(G)$. The cardinality of a largest stable set is denoted by $\alpha(G)$. A *k-coloring* is a partition of the vertices into k stable sets (these stable sets are called *color classes*). The *chromatic number* $\chi(G)$ is the smallest value of k for which there exists a k-coloring. Obviously, $\omega(G) \leq \chi(G)$ since the vertices of a clique must be in distinct color classes of a k-coloring. An *induced subgraph* of G is a graph with vertex set $S \subseteq V(G)$ and edge set comprising all the edges of G with both ends in S. It is denoted by $G(S)$. The graph $G(V(G) - S)$ is denoted by $G \setminus S$. A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraphs H of G. A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are.

A *hole* is the graph induced by a chordless cycle of length at least 4. A hole is *odd* if it contains an odd number of vertices. Odd holes are not perfect since their chromatic number is 3 whereas the size of their largest clique is 2. It is easy to check that odd holes are minimally imperfect. The complement of a graph G is the graph $\bar{G}$ with the same vertex set as G, and uv is an edge of $\bar{G}$ if and only if it is not an edge of G. It is easy to check that complements of odd holes are also minimally imperfect. In the early sixties Berge [1] proposed the *Strong Perfect Graph Conjecture*: The odd holes and their complements are the only minimally imperfect graphs. This conjecture attracted much attention over the last forty years. It was proved in May 2002 by Chudnovsky, Robertson, Seymour and Thomas [9] in a very impressive paper. Claude Berge passed away in June 2002 knowing that his famous conjecture is true.

Theorem 1.1 (Strong Perfect Graph Theorem) (Chudnovsky, Robertson, Seymour and Thomas [9]) *The only minimally imperfect graphs are the odd holes and their complements.*

In this note, we survey key aspects of the proof of the Strong Perfect Graph Theorem. A *Berge graph* is a graph that does not contain an odd hole or its complement as an induced subgraph. Clearly, every perfect graph is a Berge graph. The Strong Perfect Graph Theorem states that the converse is also true: Every Berge graph is perfect. The idea of the proof is to show that every Berge graph either falls into one of four basic classes of perfect graphs, or that it has a kind of separation that cannot occur in a minimally imperfect graph.

In [1], Berge also made a weaker conjecture, which states that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. This conjecture was proved by Lovász [24] in 1972. We give a short elegant proof due to Gasparyan [21].

Theorem 1.2 (Perfect Graph Theorem) (Lovász [24]) *Graph G is perfect if and only if graph $\bar{G}$ is perfect.*

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Proof: Lovász [25] proved the following stronger result.

Claim 1: A graph G is perfect if and only if, for every induced subgraph H, the number of vertices of H is at most $\alpha(H)\omega(H)$.

Since $\alpha(H) = \omega(\bar{H})$ and $\omega(H) = \alpha(\bar{H})$, Claim 1 implies Theorem 1.2.

Proof of Claim 1: First assume that G is perfect. Then, for every induced subgraph H, $\omega(H) = \chi(H)$. Since the number of vertices of H is at most $\alpha(H)\chi(H)$, the inequality follows.

We give a proof of the converse due to Gasparyan [21]. Assume that G is not perfect. Let H be a minimally imperfect subgraph of G and let n be the number of vertices of H. Let $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies

$$\omega = \chi(H \backslash v) \text{ for every vertex } v \in V(H)$$

$$\text{and } \omega = \omega(H \backslash S) \text{ for every stable set } S \subseteq V(H).$$

Let A_0 be an α-stable set of H. Fix an ω-coloring of each of the α graphs $H \backslash s$ for $s \in A_0$, let $A_1, \dots, A_{\alpha\omega}$ be the stable sets occuring as a color class in one of these colorings and let $\mathcal{A} := \{A_0, A_1, \dots, A_{\alpha\omega}\}$. Let $\mathbf{A}$ be the corresponding stable set versus vertex incidence matrix. Define $\mathcal{B} := \{B_0, B_1, \dots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $H \backslash A_i$. Let $\mathbf{B}$ be the corresponding clique versus vertex incidence matrix.

Claim 2: Every ω-clique of H intersects all but one of the stables sets in $\mathcal{A}$.

Proof of Claim 2: Let $S_1, \dots, S_\omega$ be any ω-coloring of $H \setminus v$. Since any ω-clique C of H has at most one vertex in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one vertex in A_0, Claim 2 follows.

In particular, it follows that $\mathbf{A}\mathbf{B}^T = J - I$ where J is the matrix filled with ones and I the identity. Since $J - I$ is nonsingular, $\mathbf{A}$ and $\mathbf{B}$ have at least as many columns as rows, that is $n \geq \alpha\omega + 1$. This completes the proof of Claim 1. □

2 Four Basic Classes of Perfect Graphs

Bipartite graphs are perfect since, for any induced subgraph H, the bipartition implies that $\chi(H) \leq 2$ and therefore $\omega(H) = \chi(H)$.

A graph L is the *line graph* of a graph G if $V(L) = E(G)$ and two vertices of L are adjacent if and only if the corresponding edges of G are adjacent.

Proposition 2.1 *Line graphs of bipartite graphs are perfect.*

Proof: If G is bipartite, $\chi'(G) = \Delta(G)$ by a theorem of König [23], where χ' denotes the edge-chromatic number and Δ the largest vertex degree.

If L is the line graph of a bipartite graph G, then $\chi(L) = \chi'(G)$ and $\omega(L) = \Delta(G)$. Therefore $\chi(L) = \omega(L)$. Since induced subgraphs of L are also line graphs of bipartite graphs, the result follows. □

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász's perfect graph theorem (Theorem 1.2) that the complements of bipartite graphs and of line graphs of bipartite graphs are perfect. This can also be verified directly, without using the perfect graph theorem. To summarize, in this section we have introduced four classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

These graphs are called *basic*.

3 2-Join, Homogeneous Pair and Skew Partition

2-Join

A graph G has a *2-join* if its vertices can be partitioned into sets V_1 and V_2, each of cardinality at least three, with nonempty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$, such that all the vertices of A_1 are adjacent to all the vertices of A_2, all the vertices of B_1 are adjacent to all the vertices of B_2 and these are the only adjacencies between V_1 and V_2. 2-joins were introduced by Cornuéjols and Cunningham [17] in 1985. They gave an $O(|V(G)|^2|E(G)|^2)$ algorithm to find whether a graph G has a 2-join.

When G contains a 2-join, we can decompose G into two blocks G_1 and G_2 defined as follows.

Definition 3.1 *If A_2 and B_2 are in different connected components of $G(V_2)$, define* block G_1 *to be $G(V_1 \cup \{p_1, q_1\})$, where $p_1 \in A_2$ and $q_1 \in B_2$. Otherwise, let P_1 be a shortest path from A_2 to B_2 and define* block G_1 *to be $G(V_1 \cup V(P_1))$. Block G_2 is defined similarly.*

Theorem 3.2 (2-Join Decomposition Theorem) (Cornuéjols and Cunningham [17]) *Graph G is perfect if and only if its blocks G_1 and G_2 are perfect.*

Corollary 3.3 *If a minimally imperfect graph G has a 2-join, then G is an odd hole.*

Proof: Since G is not perfect, Theorem 3.2 implies that block G_1 or G_2 is not perfect, say G_1. Since G_1 is an induced subgraph of G and G is minimally imperfect, it follows that $G = G_1$. Thus, since $|V_2| \geq 3$, V_2 induces a chordless path P_1. Therefore G is a minimally imperfect graph with a vertex of degree 2. It is well known that such a graph G is an odd hole [27]. □

Homogeneous Pair

The notion of homogeneous pair was introduced by Chvátal and Sbihi [5]. A graph G has a *homogeneous pair* if $V(G)$ can be partitioned into subsets A_1, A_2 and B, such that:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$.
- If a vertex of B is adjacent to a vertex of A_i then it is adjacent to all the vertices of A_i, for $i \in \{1, 2\}$.

Theorem 3.4 (Homogeneous Pair Theorem) (Chvátal and Sbihi [5]) *No minimally imperfect graph has a homogeneous pair.*

Skew Partition

A graph G has a *skew partition* if its vertices can be partitioned into four nonempty sets A, B, C, D such that there are all the possible edges between A and B and no edges from C to D. Chvátal [3] introduced skew partitions in 1985 and he conjectured that no minimally imperfect graph has a skew partition. He observed that the conjecture holds for a *star cutset*, defined to be a skew partition where $|A| = 1$.

Lemma 3.5 (Star Cutset Lemma) (Chvátal [3]) *No minimally imperfect graph has a star cutset.*

Proof: Let G_1 be the graph induced by $A \cup B \cup C$ and G_2 the graph induced by $A \cup B \cup D$. The graphs G_1 and G_2 are perfect. Let S_i be the color class of an $\omega(G)$-coloring of G_i that contains the unique node of A, for $i \in \{1, 2\}$. Then S_i meets all the $\omega(G)$-cliques of G_i, i.e. $\omega(G \setminus (S_1 \cup S_2)) < \omega(G)$. It follows that $G \setminus (S_1 \cup S_2)$ can be colored with fewer than $\omega(G)$ colors, since it is perfect. Since $S_1 \cup S_2$ is a stable set, G can be colored with $\omega(G)$ colors, a contradiction. □

Noteworthy contributions towards the skew partition conjecture were made by Hoàng [22] and Roussel and Rubio [28]. The conjecture was settled by Chudnovsky, Robertson, Seymour and Thomas [9]. They obtained it as a consequence of the Strong Perfect Graph Theorem.

Theorem 3.6 (Skew Partition Theorem) (Chudnovsky, Robertson, Seymour and Thomas [9]) *No minimally imperfect graph has a skew partition.*

In order to prove the Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour and Thomas first proved the following weaker result.

A skew partition is *balanced* if
(i) every induced path of length at least 2 in G with ends in $A \cup B$ and interior in $C \cup D$ is even, and
(ii) every induced path of length at least 2 in $\bar{G}$ with ends in $C \cup D$ and interior in $A \cup B$ is even.

Theorem 3.7 (Chudnovsky, Robertson, Seymour and Thomas [8]) *A minimally imperfect Berge graph with smallest number of vertices cannot have a balanced skew partition.*

We give the proof of Theorem 3.7. It uses Lovász's Replication Lemma [24] which we discuss next. Incidentally, the Replication Lemma was the step that Fulkerson missed in his attempt to prove the Perfect Graph Theorem. Because Fulkerson had convinced himself that it was likely to be false, he had not tried very hard to prove it. Fulkerson [20] says: "In the Spring of 1971, I received a postcard from Berge saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it."

Lemma 3.8 (Replication Lemma) (Lovász [24]) *Let G be a perfect graph and $v \in V(G)$. Create a new vertex v' and join it to v and to all the neighbors of v. Then the resulting graph G' is perfect.*

Proof: It suffices to show $\chi(G') = \omega(G')$ since, for induced subgraphs, the proof follows similarly. We distinguish two cases.

Case 1: Vertex v is contained in some maximum clique of G. Then $\omega(G') = \omega(G) + 1$. This implies $\chi(G') \leq \omega(G')$, since at most one new color is needed in G'. Clearly $\chi(G') = \omega(G')$ follows.

Case 2: Vertex v is not contained in any maximum clique of G. Consider any coloring of G with $\omega(G)$ colors and let S be the color class containing v. Then $\omega(G \setminus (S - \{v\})) = \omega(G) - 1$, since every maximum clique in G meets $S - \{v\}$. By the perfection of G, the graph $G \setminus (S - \{v\})$ can be colored with $\omega(G) - 1$ colors. Using one additional color for the vertices $(S - \{v\}) \cup \{v'\}$, we obtain a coloring of G' with $\omega(G)$ colors. $\square$

Proof of Theorem 3.7: Let G be a minimally imperfect Berge graph with smallest number of vertices. Suppose that G has a balanced skew partition A, B, C, D. By the Star Cutset Lemma 3.5, each of A, B, C, D has cardinality at least two. Let G' be the graph obtained from G by adding a vertex v adjacent to all the vertices of A and to no other vertex of G. If G' contains an odd hole, then G has an odd path contradicting (i) in the definition of a balanced skew partition. Similarly, if $\bar{G}'$ contains an odd hole, (ii) is contradicted. Therefore G' is a Berge graph. Now consider $G_1 = G' \setminus D$ and $G_2 = G' \setminus C$. For $i \in \{1, 2\}$, the graph G_i is perfect since it is Berge and has fewer vertices than G. Replicate vertex v in G_i so that v belongs to a clique of size $\omega(G)$. By the Replication Lemma 3.8, the resulting graph R_i is perfect. Consider $\omega(G)$-colorings of R_1 and R_2 respectively. Both colorings have the same number of colors in A and assume w.l.o.g. that these colors are $1, 2, \ldots, k$. Let K be the subgraph of G induced by the vertices with colors $1, 2, \ldots, k$ and let H be the subgraph of G induced by the vertices with other colors. Since every $\omega(G)$-clique of G is in $G \setminus D$ or $G \setminus C$, the largest clique in K has size k and the largest clique in H has size $\omega(G) - k$. The graphs H and K are perfect since they are proper subgraphs of G. Color K with k colors and H with $\omega(G) - k$ colors. Now G is colored with $\omega(G)$ colors, a contradiction to the assumption that G is minimally imperfect. $\square$

Theorem 3.7 was presented in September 2001 at a workshop in Princeton. As the next step towards Theorem 3.6, Chudnovsky and Seymour obtained the following theorem in January 2002.

Theorem 3.9 (Chudnovsky and Seymour [10]) *A minimally imperfect Berge graph with smallest number of vertices cannot have a skew partition.*

4 Decomposition of Berge Graphs

Conforti, Cornuéjols and Vušković proposed the following approach to solving the Strong Perfect Graph Conjecture.

Conjecture 4.1 (Conforti, Cornuéjols and Vušković (2001)) **(Decomposition Conjecture)** *Every Berge graph G is basic or has a skew partition, or G or $\bar{G}$ has a 2-join.*

Chudnovsky, Robertson, Seymour and Thomas proved the following variation of this conjecture.

Theorem 4.2 (Chudnovsky, Robertson, Seymour and Thomas [9]) **(Decomposition Theorem)** *Every Berge graph G is basic or has a skew partition or a homogeneous pair, or G or $\bar{G}$ has a 2-join.*

This theorem implies the Strong Perfect Graph Theorem. Indeed, suppose that the Decomposition Theorem holds and that there exists a minimally imperfect graph G distinct from an odd hole or its complement. Choose G with the smallest number of vertices. G cannot have a skew partition by Theorem 3.9. G cannot have a homogeneous pair by Theorem 3.4. Neither G nor $\bar{G}$ can have a 2-join by Corollary 3.3. Since G is a Berge graph, G must be basic by the Decomposition Theorem. Therefore G is perfect, a contradiction.

Theorem 4.2 was already known to hold in several special cases. For example, it was known when G is a Meyniel graph (Burlet and Fonlupt [2] in 1984), when G is claw-free (Chvatal and Sbihi [6] in 1988 and Maffray and Reed [26] in 1999), diamond-free (Fonlupt and Zemirline [19] in 1987), bull-free (Chvátal and Sbihi [5] in 1987), or dart-free (Chvátal, Fonlupt, Sun and Zemirline [4] in 2000). All these results involve special types of skew partitions (such as star cutsets) and, in some cases, homogeneous pairs [5]. A special case of 2-join called augmentation of a flat edge appears in [26]. In 1999, Conforti and Cornuéjols [13] used more general 2-joins to prove Conjecture 4.1 for WP-free Berge graphs, a class of graphs that contains all bipartite graphs and all line graphs of bipartite graphs. [13] was the precursor of a sequence of decomposition results involving 2-joins. The following result was obtained in February 2001.

Theorem 4.3 (Conforti, Cornuéjols and Vušković [14]) *A square-free Berge graph is bipartite, the line graph of a bipartite graph, or has a 2-join or a star cutset.*

A breakthrough occured in September 2001 when Chudnovsky, Robertson, Seymour and Thomas announced that they could prove the Decomposition Conjecture in the following important special case.

Theorem 4.4 (Chudnovsky, Robertson, Seymour and Thomas [8]) *If G is a Berge graph that contains the line graph of a bipartite subdivision of a 3-connected graph, then G has a balanced skew partition, or G or $\bar{G}$ has a 2-join or is the line graph of a bipartite graph.*

Given two vertex disjoint triangles a_1, a_2, a_3 and b_1, b_2, b_3, a *subdivided prism* is a graph induced by three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, at least one of which has length greater than one, such that P^1, P^2, P^3 have no common vertices and the only adjacencies between the vertices of distinct paths are the edges of the two triangles. The next result, obtained in January 2002, is a real tour-de-force and a key step in the proof of the Strong Perfect Graph Theorem. In particular, it was needed to prove Theorem 3.9.

Theorem 4.5 (Chudnovsky and Seymour [10]) *If G is a Berge graph that contains a subdivided prism, then G is the line graph of a bipartite graph or G has a balanced skew partition or a homogeneous pair, or G or $\bar{G}$ has a 2-join.*

A *wheel* (H, v) consists of a hole H together with a vertex v, called the *center*, with at least three neighbors in H. If v has k neighbors in H, the wheel is called a *k-wheel*. A *line wheel* is a 4-wheel (H, v) that contains exactly two triangles and these two triangles have only the center v in common. A *twin wheel*

is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the vertices of H. A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types. These concepts were first introduced in [13]. The following theorem, obtained in May 2002, generalizes an earlier result of Zambelli presented in September 2001 and of Thomas [29].

Theorem 4.6 (Conforti, Cornuéjols and Zambelli [16]) *If G is a Berge graph that contains no proper wheel, subdivided prism or their complements, then G is basic or has a skew partition.*

The last step in proving the Strong Perfect Graph Theorem is the following difficult theorem, also obtained in May 2002.

Theorem 4.7 (Chudnovsky and Seymour [11]) *If G is a Berge graph that contains a proper wheel, but no subdivided prism or its complement, then G has a skew partition, or G or $\bar{G}$ has a 2-join.*

Theorems 4.5, 4.6 and 4.7 imply the Decomposition Theorem 4.2, and therefore the Strong Perfect Graph Theorem. A monumental paper containing these results is now available [9].

Conforti, Cornuéjols and Vušković [15] proved a weaker version of the Decomposition Conjecture where "skew partition" is replaced by "double star cutset". A *double star* is a vertex set S that contains two adjacent vertices u, v and a subset of the vertices adjacent to u or v. Clearly, if G has a skew partition, then G has a double star cutset: Take $S = A \cup B$, $u \in A$ and $v \in B$. Although the decomposition result in [15] is weaker than Conjecture 4.1 for Berge graphs, it holds for a larger class of graphs than Berge graphs: By changing the decomposition from "skew partition" to "double star cutset", the result can be obtained for all odd-hole-free graphs instead of just Berge graphs.

Theorem 4.8 (Conforti, Cornuéjols and Vušković [15]) *If G is an odd-hole-free graph, then G is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or G has a double star cutset or a 2-join.*

Theorem 4.8 was used by Cornuéjols, Liu and Vušković [18] to construct a polynomial time recognition algorithm for perfect graphs. Independently, Chudnovsky and Seymour [12] found a different algorithm for perfect graph recognition which does not use decomposition. Both algorithms [12], [18] build on the same companion paper [7] which performs a certain "cleaning" step in polynomial time.

A useful tool for studying Berge graphs is due to Roussel and Rubio [28]. This lemma was proved independently by Chudnovsky, Robertson, Seymour and Thomas [8], who popularized it and named it *The Wonderful Lemma*. It is used repeatedly in the proofs of Theorems 4.4-4.7.

Lemma 4.9 (The Wonderful Lemma) (Roussel and Rubio [28]) *Let G be a Berge graph and assume that $V(G)$ can be partitioned into a set S and an odd chordless path $P = u, u', \ldots, v', v$ of length at least 3 such that u, v are both adjacent to all the vertices in S and $\bar{G}(S)$ is connected. Then one of the following holds:*

(i) An odd number of edges of P have both ends adjacent to all the vertices in S.

(ii) P has length 3 and $\bar{G}(S \cup \{u', v'\})$ contains an odd chordless path between u' and v'.

(iii) P has length at least 5 and there exist two nonadjacent vertices x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a path.

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